Physics of the Dangling Stick

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The Dangling Stick is a massless rigid stick with a point mass on each end, see figure 1. One end of the stick is attached to a spring, and gravity acts. The equations of motion are derived by the Lagrangian method.

We will use subscript 1 for the mass at the spring-stick intersection, and subscript 2 for the mass at the free end of the stick.

The three independent variables are:

θ	angle of spring (0 = vertical downwards)
ϕ	angle of stick (0 = vertical downwards)
r	length of spring

The cartesian position of the point masses is given by R_1, R_2.

The other constants used are listed below.

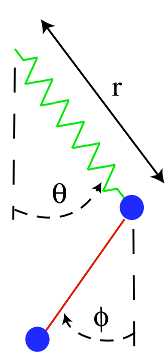

Figure 1: Variables for Dangling Stick

m_1, m_2	mass of the point masses
L	length of stick
k	spring stiffness constant
h	spring rest length
g	gravity constant

1 Kinematics

First we calculate the kinematics. That is, we find expressions for the positions R_1, R_2 of the two masses in terms of the above variables. Note that there is no information about forces here, kinematics is mostly just geometry.

$$R_1 = \{r \sin\theta, -r\cos\theta\} \tag{1}$$
$$R_2 = \{r\sin\theta + L\sin\phi, -(r\cos\theta + L\cos\phi)\} \tag{2}$$

The two components correspond to $\{x, y\}$ respectively. Next we differentiate to get the velocities of the masses.

$$\dot{R}_1 = v_1 = \{r\dot{\theta}\cos\theta + \dot{r}\sin\theta, -\dot{r}\cos\theta + r\dot{\theta}\sin\theta\} \tag{3}$$
$$\dot{R}_2 = v_2 = \{r\dot{\theta}\cos\theta + \dot{r}\sin\theta + L\dot{\phi}\cos\phi, -\dot{r}\cos\theta + r\dot{\theta}\sin\theta + L\dot{\phi}\sin\phi\} \tag{4}$$

2 Energy and the Lagrangian

Next we get separate expressions for the kinetic energy T and potential energy V of the system. The difference of the two is the Lagrangian $\mathcal{L}$.

The kinetic energy is given by the standard formula $\frac{1}{2}mv^2$ but here we keep in mind that v is a vector and so use the dot product.

$$T = \frac{m_1}{2} v_1 \cdot v_1 + \frac{m_2}{2} v_2 \cdot v_1 \tag{5}$$

The potential energy is from the spring energy and the gravitational potential of the two point masses.

$$V = \frac{k}{2}(r-h)^2 - m_2 g(r\cos\theta + L\cos\phi) - m_1 g r\cos\theta \tag{6}$$

The Lagrangian is the difference between the two

$$\mathcal{L} = T - V \tag{7}$$

3 Lagrangian Equations

Now we evaluate the Lagrangian equation once for each of the three independent variables.

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} \tag{8}$$

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} \tag{9}$$

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} \tag{10}$$

4 Equations of Motion

The three Lagrangian equations can now be solved to find expressions for the second derivatives $\ddot{\theta}, \ddot{\phi}, \ddot{r}$. After using a computer algebra program such as *Mathematica* we find the following equations of motion.

$$\ddot{\theta} = \frac{1}{2m_1(m_1+m_2)r}\Big(km_2(h-r)\sin(2\theta - 2\phi) - 2gm_1(m_1+m_2)\sin\theta - 4m_1(m_1+m_2)\dot{r}\dot{\theta} + 2Lm_1m_2\dot{\phi}^2\sin(\theta - \phi)\Big) \tag{11}$$

$$\ddot{r} = \frac{1}{2m_1(m_1+m_2)}\Big(2khm_1 + khm_2 - 2km_1r - km_2r + 2m_1^2r\dot{\theta}^2 + 2m_1m_2r\dot{\theta}^2 + 2gm_1(m_1+m_2)\cos\theta + 2Lm_1m_2\dot{\phi}^2\cos(\theta-\phi) - km_2(h-r)\cos(2\theta - 2\phi)\Big) \tag{12}$$

$$\ddot{\phi} = \frac{k(h-r)\sin(\theta-\phi)}{Lm_1} \tag{13}$$

These equations are of the form needed to use with the Runge-Kutta algorithm to numerically solve the equations.